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A conservative split-explicit integration scheme suitable for forecast and climate models

By M. J. P. CULLEN[1] and T. DAVIES[2]

[1] *Meteorological Office, Bracknell*

[2] *European Centre for Medium Range Weather Forecasts, Reading.*

SUMMARY

A split-explicit finite difference scheme is developed which combines the accuracy and economy required for numerical weather prediction with the conservation properties required for climate change experiments. Results are presented showing that the aims of the scheme are achieved.

1. INTRODUCTION

While spectral methods are in almost universal use for the global atmospheric models used for weather forecasting and climate research, there is still an interest in developing finite difference methods suitable for these applications. This is because, as resolution increases, the cost of the spectral method may increase faster than that of finite difference methods and also, as interest focusses on local behaviour in the results, finite difference methods may be found better at representing local, rather discontinuous behaviour.

A very efficient finite difference scheme developed for use in forecast models is the split-explicit method, Gadd (1978a). Gadd combined this method with an accurate version of the Lax-Wendroff advection scheme, Gadd (1978b). Mesinger (1981) used the split-explicit method together with the Heun advection scheme, which avoids the complexity of time-staggering. Both schemes have been used successfully in operational models for a number of years; the former has been used in the U.K. Meteorological Office global operational model since 1982.

In climate research an important requirement is to conserve mass-weighted temperature and moisture, so that the heat and moisture budgets can be accurately calculated and their effect in various climate change scenarios assessed. Neither of the above split-explicit schemes do this when written in the standard sigma or hybrid sigma/pressure vertical coordinate systems. The difficulty is that the vertical advection of potential temperature forms part of the gravity wave dynamics and has to be treated separately from the horizontal advection. It may also be important to conserve quadratic quantities, such as the enstrophy, to ensure satisfactory behaviour in long-term integrations. Mesinger (1981) has shown

that these higher order conservation laws can be respected in split-explicit finite difference schemes.

In this paper, a split-explicit scheme which conserves mass weighted temperature and moisture is developed. This is done by treating only the vertical advection of a basic state temperature with a short timestep, and treating the remainder of the advection with a conservative scheme and a long timestep. The technique is exactly parallel to the split used in semi-implicit models, Simmons et al. (1978). It is also necessary to ensure that a finite difference flux conservation law can be obtained by combining the advection scheme with the continuity equation in finite difference form. This is done by using the Heun scheme, since this combination is difficult to achieve with the time-staggered Lax-Wendroff scheme.

2. THE FORECAST EQUATIONS

To simplify the presentation, the scheme is described in horizontal Cartesian coordinates (x, y). When used in a global model the application on a latitude longitude grid in spherical polar coordinates is straightforward. Careful attention has to be paid to the method of filtering near the poles to ensure that the conservation properties are retained, the method required is discussed at the end of section 3. The construction of conservation laws depends on the continuity equation whose form depends on the choice of vertical coordinate. We use the hybrid coordinate system described by Simmons and Burridge (1981) which combines the advantages of the terrain-following sigma coordinate system near the surface and the pressure system in the stratosphere, thereby reducing the error in the pressure gradient term. Only the adiabatic equations are presented, since moisture conservation is assured by using the same advection scheme as for the perturbation potential temperature.

Define a vertical coordinate $\eta = \eta(p, p_*)$, where $\eta(0, p_*)=0$ and $\eta(p_*, p_*)=1$. The equations are then

$$\partial u/\partial t + u\partial u/\partial x + v\partial u/\partial y + \dot{\eta}\partial u/\partial \eta + \partial \Phi/\partial x + (RT/p)\partial p/\partial x - fv = F_u \quad (1)$$

$$\partial v/\partial t + u\partial v/\partial x + v\partial v/\partial y + \dot{\eta}\partial v/\partial \eta + \partial \Phi/\partial y + (RT/p)\partial p/\partial y + fu = F_v \quad (2)$$

$$\partial \theta/\partial t + u\partial \theta/\partial x + v\partial \theta/\partial y + \dot{\eta}\partial \theta/\partial \eta = F_\theta \quad (3)$$

$$\partial/\partial \eta(\partial p/\partial t) + \partial/\partial x(u\partial p/\partial \eta) + \partial/\partial y(v\partial p/\partial \eta)\} + \partial/\partial \eta(\dot{\eta}\partial p/\partial \eta) = 0 \quad (4)$$

The quantities F_u, F_v, F_θ represent source terms, and also include any diffusion required for computational reasons. The vertical boundary conditions are:

$$\dot{\eta} = 0 \text{ at } \eta = 0, 1. \quad (5)$$

Integrating (4) in the vertical from η=0 to 1 gives:

$$\partial p_*/\partial t = -\int^1_0 [\partial/\partial x(u\partial p/\partial \eta) + \partial/\partial y(v\partial p/\partial \eta)]d\eta \quad (6)$$

Integrating (4) from η=0 to η gives:

$$\dot{\eta}\partial p/\partial \eta = -\partial p/\partial t - \int^\eta_0 [\partial/\partial x(u\partial p/\partial \eta) + \partial/\partial y(v\partial p/\partial \eta)]d\eta \quad (7)$$

The hydrostatic relation is given by:

$$\partial \Phi/\partial \eta = -(RT/p)\partial p/\partial \eta = -c_p\theta\partial \Pi/\partial \eta, \quad (8)$$

where $\Pi=(p/1000)^\kappa$.

3. THE INTEGRATION SCHEME

The variables are held on the Arakawa 'B' grid . The variables u, v, θ and Φ are held at levels η_k, where k is the vertical grid-length index, while $\dot{\eta}$ is held at the intermediate levels $\eta_{k+\frac{1}{2}}$. The lower boundary is k=½ and the upper boundary k=TOP+½.. The pressure is defined at intermediate levels by

$$p_{k+\frac{1}{2}} = A_{k+\frac{1}{2}} + B_{k+\frac{1}{2}} p_* , \qquad (9)$$

where $A_{k+\frac{1}{2}}$ and $B_{k+\frac{1}{2}}$ are specified constants. Thus

$$(\partial p/\partial p_*)_{k+\frac{1}{2}} = B_{k+\frac{1}{2}} \qquad (10)$$

and

$$\Delta p_k = (A_{k+\frac{1}{2}} - A_{k-\frac{1}{2}}) + (B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}}) p_* . \qquad (11)$$

Note that this definition makes Δp_k negative, since k increases with physical height. In the split explicit integration scheme, the solution procedure is split into two parts, called the 'adjustment' and 'advection' steps. The adjustment timestep is written as δt, the advection timestep as Δt. In the former, the pressure, temperature, and wind fields are updated using the pressure gradient and the Coriolis terms, and the vertical advection of the basic state potential temperature. Only the final updated values of surface pressure and horizontal wind are used in the next step. The average horizontal wind from the adjustment step is used to define the horizontal advection in the advection step, and, via the continuity equation, the vertical advection. This procedure is needed to ensure conservation. All advection increments except the vertical advection of basic state potential temperature are then calculated in the advection step, together with any horizontal diffusion.

The standard finite difference notation

$$\delta_x X = (X(x+\tfrac{1}{2}\Delta x) - X(x-\tfrac{1}{2}\Delta x))/\Delta x$$

$$\overline{X}^{x} = \tfrac{1}{2}(X(x+\tfrac{1}{2}\Delta x) + X(x-\tfrac{1}{2}\Delta x))$$

is used.

(a) *The adjustment step*

This uses the 'forward-backward' scheme in which a forward step is used for the u and v equations, and the new values of these variables are then used in the p_* and θ equations. The 'forward' part of the integration scheme is:

$$u_k^{n+1} = u_k^n + \delta t\{\tfrac{1}{2}f(v_k^n + v_k^{n+1}) - \left(\delta_x\Phi_k^n + \frac{C_p\theta_k}{(\varkappa+1)}\,\delta_x\left[\frac{\Pi_{k+\frac{1}{2}}p_{k+\frac{1}{2}} - \Pi_{k-\frac{1}{2}}p_{k-\frac{1}{2}}}{\Delta p_k}\right]^n_x\right), \tag{12}$$

$$v_k^{n+1} = v_k^n - \delta t\{\tfrac{1}{2}f(u_k^n + u_k^{n+1}) + \left(\delta_y\Phi_k^n + \frac{C_p\theta_k}{(\varkappa+1)}\,\delta_y\left[\frac{\Pi_{k+\frac{1}{2}}p_{k+\frac{1}{2}} - \Pi_{k-\frac{1}{2}}p_{k-\frac{1}{2}}}{\Delta p_k}\right]^n_y\right), \tag{13}$$

Equations (12) and (13) can be arranged to allow explicit integration. The hydrostatic equation is approximated by

$$\Phi_k = \Phi_* + \sum_{m=1}^{k-1} C_p\theta_m(\Pi_{m+\frac{1}{2}} - \Pi_{m-\frac{1}{2}}) + C_p\theta_k(\Pi_k - \Pi_{k-\frac{1}{2}}) - \frac{(\Pi_{k-\frac{1}{2}}p_{k-\frac{1}{2}} - \Pi_{k+\frac{1}{2}}p_{k+\frac{1}{2}})}{(\varkappa+1)\Delta p_k}\Big). \tag{14}$$

The special form of the last term is chosen to ensure angular momentum conservation.

The 'backward' part of the integration scheme is given by

$$p_*^{n+1} = p_*^n - \delta t\sum_{m=1}^{TOP} D_m^{n+1} \tag{15}$$

$$\theta^{n+1} = \theta^n - \frac{\delta t}{2(\Delta p)_k^n}\left[\left(\eta\frac{\partial p}{\partial \eta}\right)^{n+1}_{k+\frac{1}{2}}(\theta_{Rk+1} - \theta_{Rk}) + \left(\eta\frac{\partial p}{\partial \eta}\right)_{k-\frac{1}{2}}(\theta_{Rk} - \theta_{Rk-1})\right] \tag{16}$$

where $\theta_R(\eta)$ is a basic state profile of θ. As discussed by Simmons et al. (1978), this must be carefully chosen to ensure computational stability.

The standard choice, used here, is an isothermal basic state with temperature 300°K and surface pressure 1000hpa. The form of these equations ensures that mass and mass-weighted potential temperature are conserved; in particular the integral of $\Delta p^{n}\theta^{n+1}$ equals that of $\Delta p^{n}\theta^{n}$.

$$\left(\frac{\partial \eta}{\partial p}\right)_{k+\frac{1}{2}} = \left(\frac{\partial \eta}{\partial p_*}\right)_{k+\frac{1}{2}} \sum_{m=1}^{TOP} D_m - \sum_{m=1}^{k} D_m \qquad (17)$$

$$D_m = [\delta_x(u_m \overline{\Delta p_m}^{xy}) + \delta_y(v_m \overline{\Delta p_m}^{xy})]^{y}_{x}. \qquad (18)$$

In order to ensure that θ is conserved under advection, it is necessary that all advection is done by a three-dimensional velocity field which satisfies the continuity equation. The average values of $u_m\Delta p_m$ and $v_m\Delta p_m$ over the adjustment steps must be saved for use in the advection step.

(b) *The advection step*

The basic state θ_R is first subtracted from θ to give θ'. The Heun advection scheme is used, as in Mesinger (1981). Experiments within the split-explicit model described by Bell and Dickinson (1987) have shown that it is more stable than the Lax-Wendroff scheme used in that model, even though it has growing eigensolutions of order $(1+O(\Delta t^4))$. The scheme can be corrected to remove this instability, but in practice it is found that this correction is submerged in the diffusion required for other reasons, and the correction is therefore not used here. The scheme has two steps. The advecting velocity for both is the average value saved from the adjustment steps.

Define

$$\underline{U}_k = (U_k, V_k) = (u_k \overline{\Delta p_k}^{xy}, v_k \overline{\Delta p_k}^{xy}), \qquad (19)$$

as saved from the adjustment steps. Define

$$\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}} \equiv E_{k+\frac{1}{2}}, \tag{20}$$

where $E_{k+\frac{1}{2}}$ is calculated from the finite difference formulae (17) and (18). The finite difference equations for the first advection step are then:

$$\Delta p_k^n \theta_k'^{\#} = \Delta p_k^n \theta_k'^{n} -$$

$$\Delta \{(1+\nu)\overline{\overline{U_m}^{y}}^{x}\delta_x\theta_k' - \nu\overline{\overline{U_m}^{y}}^{3x}\delta_x\theta_k' + (1+\nu)\overline{\overline{V_m}^{x}}^{y}\delta_y\theta_k' - \nu\overline{\overline{V_m}^{x}}^{3y}\delta_y\theta_k'\} \tag{21}$$

$$- \tfrac{1}{2}\Delta t\{E_{k+\frac{1}{2}}(\theta'_{k+1}-\theta'_k) + E_{k-\frac{1}{2}}(\theta'_k-\theta'_{k-1})\},$$

$$\overline{(\Delta p)_k^n}^{xy} u_k^{\#} = \overline{(\Delta p)_k^n}^{xy} u_k^n -$$

$$\{(1+\nu)\overline{\overline{U_m}^{y}}^{x}\delta_x u_k - \nu\overline{\overline{U_m}^{y}}^{3x}\delta_x u_k + (1+\nu)\overline{\overline{V_m}^{x}}^{y}\delta_y u_k - \nu\overline{\overline{V_m}^{x}}^{3y}\delta_y u_k\}, \tag{22}$$

$$- \tfrac{1}{2}\Delta t\{\overline{E_{k+\frac{1}{2}}}^{xy}(u_{k+1}-u_k) + \overline{E_{k-\frac{1}{2}}}^{xy}(u_k-u_{k-1})\},$$

with a similar equation for v.

Note that in the scheme of Gadd (1978b), higher accuracy is achieved without requiring the timestep to be reduced by modifying the second step of the Lax-Wendroff scheme. In the Heun scheme, it is necessary to use the same finite difference approximation in both steps, or else there is an $O(\Delta t^2)$ instability. The value $\nu=1/6$ in equations (21) to (22) gives fourth order accuracy, but will increase the squared amplification rate of the growing solution from $(1+\frac{1}{4}\xi^4)$ to $(1+\frac{1}{4}\xi_1^4)$ where ξ is the Courant number and $\xi_1=1.37\xi$. This will reduce the maximum timestep that can safely be used. A fixed value must be used for ν to allow conservation, but for applications where this is not important the choice $\nu=1/6(1-\xi^2)$ should avoid the need to reduce the timestep.

The second advection step can be written:

$$(\Delta p)_k^{n+1}\,\theta_k^{'n+1} = \{\tfrac{1}{2}(\Delta p)_k^{n}(\theta_k^{'n} - \theta_k^{'\#}) + \tfrac{1}{2}(\Delta p)_k^{n+1}(\theta_k^{'n} + \theta_k^{'\#})\} - \tfrac{1}{2}\Delta t\,(\underline{U}^{'n}\cdot\nabla\theta_k^{'\#} + \underline{U}^{'\#}\cdot\nabla\theta_k^{'n}). \quad (23)$$

where $\underline{U}$ is the three-dimensional velocity vector. The equation for u is

$$(\Delta p)_{xy}^{n+1}\,u_k^{n+1} = \{\tfrac{1}{2}(\Delta p)_{xy}^{n}(u_k^{n} - u_k^{\#}) + \tfrac{1}{2}(\Delta p)_{xy}^{n+1}(u_k^{n} + u_k^{\#})\} - \tfrac{1}{2}\Delta t\,(\underline{U}^{n}\cdot\nabla u_k^{\#} + \underline{U}^{\#}\cdot\nabla u_k^{n}) \quad (24)$$

with a similar equation for v. The form of equations (23) and (24) ensures conservation under time differencing.

(c) *Fourier filtering*

When this finite difference scheme is used in a global model on a latitude-longitude grid, some form of filtering is needed at high latitudes to avoid the need for a very short timestep. It is necessary to ensure that global conservation properties are not affected by the filtering. Mass-weighted increments to θ, and mass weighted velocity fields $\Delta p(u, v)$ are therefoire filtered. Filtering mass-weighted velocity fields before the update to p_* removes the need to filter p_* and θ increments after the adjustment steps, so that the conservation proofs of section 4 do not have to consider the effect of filtering. This strategy also avoids the problem of filtering fields which vary rapidly along a model coordinate surface.

4. CONSERVATION PROPERTIES

(a) *Angular momentum conservation*

In this section we show that the angular momentum conservation property demonstrated by Simmons and Burridge (1981) with temperature as a model variable is retained in the present formulation. The requirement is that the pressure gradient term can only change the angular momentum

through the surface torque. This means that we must be able to write the approximation to the pressure gradient term in the model which is

$$\sum_{m=1}^{TOP} \left(\frac{\partial \Phi_m}{\partial x} + C_p \theta \frac{\partial \Pi_m}{\partial x}\right) \Delta p_m \,, \tag{25}$$

in the form

$$\frac{\partial}{\partial x}\left(\sum_{m=1}^{TOP} \Phi_m \Delta p_m\right) - \Phi_* \frac{\partial p_*}{\partial x} \,. \tag{26}$$

The first term in (26) integrates to zero and the second integrates to the surface torque. This requirement determines how the terms in (26) have to be calculated at level m. Write (14) as

$$\Phi_k = \Phi_* + \sum_{m=1}^{k-1} C_p \theta_m \Delta \Pi_m + \alpha_k C_p \Pi_k \theta_k \,. \tag{27}$$

Choose α_m such that

$$\alpha_m \Pi_m = \Pi_{m-\frac{1}{2}} + \frac{\Delta(\Pi p)_m}{(x+1)\Delta p} \,. \tag{28}$$

Then after some algebra, the summand in the second term of (25) can be written

$$\frac{C_p \theta_m}{(x+1)} \frac{\partial}{\partial x} \frac{(\Delta(\Pi p)_m)}{\Delta p_m} \,. \tag{29}$$

The expression (29) is then approximated by spatial finite differences and used to approximate the second term of (25) in the equation of motion. The above argument can be carried through in finite differences, provided that the approximation used is

$$\frac{C_p \theta_m}{(x+1)} \left[\delta_x \frac{(\Delta(\Pi p)_m)}{\Delta p_m}\right], \tag{30}$$

as used in equations (12) and (13).

(b) *Conservation of first moments*

The requirement is that the global mass-weighted mean of all advected quantities is conserved. The proof is written out only for meridional advection of θ and u, since this covers all the possible staggerings of variables that occur in the other cases. Combining (17-20) gives the continuity equation in the form:

$$E_{k+\frac{1}{2}} = \frac{(\partial p)}{(\partial p_*)_{k+\frac{1}{2}}} \sum_{m=1}^{TOP} D_m - \sum_{m=1}^{k} D_m . \qquad (31)$$

where

$$D_m = \delta_y \bar{V}_m^x . \qquad (32)$$

A simple second order forward update of θ by meridional advection, and advection by the vertical motion associated with the meridional motion, is given by

$$\overset{+}{\Delta p}_k \overset{+}{\theta}_k = \Delta p_k \theta_k - \Delta t \left[\overline{\bar{V}_k^x \delta_y \theta_k}^y + \right.$$

$$\left. \tfrac{1}{2}\left(E_{k+\frac{1}{2}}(\theta_{k+1}-\theta_k) + E_{k-\frac{1}{2}}(\theta_k-\theta_{k-1})\right)\right] . \qquad (33)$$

The update of p_* can be written

$$\overset{+}{p}_* = p_* - \Delta t \sum_{m=1}^{TOP} D_m , \qquad (34)$$

because of the definition of V_m as the average over the adjustment steps. Equation (10) can be used to rewrite (33) as

$$E_{k+\frac{1}{2}} - E_{k-\frac{1}{2}} = \Delta B_k (\overset{+}{p}_* - p_*)/\Delta t - D_k . \qquad (35)$$

Multiplying (35) by θ_k, substituting for Δp_k using (11), and adding to (33) gives

$$(\Delta A_k + \Delta B_k \overset{+}{p}_*)(\overset{+}{\theta}_k - \theta_k) + \overset{+}{\theta}_k (\overset{+}{p}_* - p_*)\Delta B_k =$$

$$-\Delta t \left[\overline{\bar{V}_k^x \delta_y \theta_k}^y + \theta_k \overline{\delta_y \bar{V}_k^x}^x + \right.$$

$$\left. \tfrac{1}{2}\left(E_{k+\frac{1}{2}}(\theta_{k+1} - \theta_k) + (E_{k-\frac{1}{2}}(\theta_k - \theta_{k-1}) + 2\theta_k (E_{k+\frac{1}{2}} - E_{k-\frac{1}{2}})\right)\right] . \qquad (36)$$

This reduces to

$$(\Delta p_k \theta_k)^{+} - \Delta p_k \theta_k = -\Delta t\,[\delta_y \overline{V}_k^{x} \overline{\theta}_k^{y} +$$

$$\tfrac{1}{2}\{E_{k+1/2}(\theta_{k+1} + \theta_k) - E_{k-1/2}(\theta_k + \theta_{k-1})\}], \tag{37}$$

which gives the desired conservation integral when integrated over y.

The update of u by meridional advection and advection by the associated part of the vertical motion is given by

$$\Delta p_k^{+} u_k^{+} = \Delta p_k^{+} u_k - \Delta t\,[\overline{\overline{V}_k^{xy}}^{x}\,\overline{\delta_y u_k}^{y} + \tfrac{1}{2}\{\overline{E}_{k+1/2}^{xy}(u_{k+1}-u_k) +$$

$$\overline{E}_{k-1/2}^{xy}(u_k - u_{k-1})\}]. \tag{38}$$

Multiplying $\overline{(35)}^{xy}$ by u_k and adding gives

$$\overline{\Delta p_k^{+}}^{xy}\, u_k - \overline{\Delta p_k}^{xy}\, u_k = -\Delta t\,[\{(\overline{\overline{V}_k^{xy}}^{x}\,\overline{\delta_y u_k}^{y} + u_k \overline{\delta_y V_k}^{xy}) +$$

$$\tfrac{1}{2}\{\overline{E}_{k+1/2}^{-\lambda\phi}(u_{k+1} - u_k) + \overline{E}_{k-1/2}^{\lambda\phi}(u_k - u_{k-1}) + 2u_k(\overline{E}_{k+1/2}^{\lambda\phi} - \overline{E}_{k-1/2}^{\lambda\phi})\}]. \tag{39}$$

The right hand side of (39) reduces to

$$-\Delta t\,[\delta_y(\overline{\overline{V}_k^{xy}}^{x}\,\overline{u_k}^{y}) + \tfrac{1}{2}\{\overline{E}_{k+1/2}^{xy}(u_{k+1}+u_k) - \overline{E}_{k-1/2}^{xy}(u_k+u_{k-1})\}]. \tag{40}$$

This is in conservation form.

Now consider the fourth order terms in (21) and (22). Conservation cannot be achieved if ν is a function of ξ, as may be necessary to avoid reducing the timestep. Suppose that ν is a constant. The terms

$$(1+\nu)\overline{\overline{U}_k^{y}}^{x}\,\delta_x\theta_k - \nu\overline{\overline{U}_k^{y}}^{3x}\,\delta_x\theta_k$$

can be expanded as

$$(1+\nu)\overline{U}_k^{y}(x+\tfrac{1}{2}\Delta x)(\theta_k(x+\Delta x)-\theta_k(x)) -$$

$$\nu\overline{U}_k^{y}(x+3/2\Delta x)(\theta_k(x+2\Delta x)-\theta_k(x+\Delta x)), \tag{41}$$

with symmetrical terms in $-\Delta x$. These terms cancel with contributions from $\theta_k^+(x+\Delta x)$ and $\theta_k^+(x-\Delta x)$ when $p_*^+\theta_k^+$ is summed over x to give the required conservation.

(c) *Conservation of second moments*

We first demonstrate that the integral of $\Delta p\theta^2$ is conserved using the second order accurate approximation to the advection terms. This will give a reasonable approximation to the true Lagrangian conservation property of θ, which requires that all moments are conserved by advection by an incompressible flow. Multiply (35) by θ_k^2 and add to (33) multiplied by $2\theta_k$:

$$2\Delta p_k \theta_k [\theta_k^+ - \theta_k] + \theta_k^2 (p_*^+ - p_*)\Delta B_k =$$

$$-\Delta t[\{2\theta_k \underline{V}_k^x \delta_y \theta_k + \theta_k \delta_y \underline{\underline{V}}_k^x\} + \{\theta_k E_{k+\frac{1}{2}}(\theta_{k+1} - \theta_k) +$$

$$\theta_k E_{k-\frac{1}{2}}(\theta_k - \theta_{k-1}) + \theta_k^2 (E_{k+\frac{1}{2}} - E_{k-\frac{1}{2}})\}]. \qquad (42)$$

The left hand side is a discrete approximation to

$$\Delta p_k \frac{\partial}{\partial t}(\theta_k^2) + \theta_k^2 \frac{\partial}{\partial t}\Delta p_k. \qquad (43)$$

However, it cannot be written as exact conservation of $\Delta p_k \theta_k^2$. The right hand side becomes

$$-\Delta t[\delta_y \underline{V}_k^x (2(\theta_k^y)^2 - \theta_k^{\underline{\underline{y}}2}) + \theta_k \theta_{k+1} E_{k+\frac{1}{2}} - \theta_k \theta_{k-1} E_{k-\frac{1}{2}}]. \qquad (44)$$

This is in conservation form. In order to achieve quadratic conservation with the fourth order terms included, the E_k's must be redefined (M. Fisher, private communication). The resulting scheme is rather less accurate because it uses a broader stencil of gridpoints. A similar argument demonstrates conservation of $u^2\Delta p$ by meridional advection.

5. RESULTS

Only sufficient results are stated here to show that the scheme achieves its aims. The performance of the scheme in a real data integration depends on the representation of physical processes as well as the dynamical scheme, and the performance of the Heun advection scheme in idealised problems has been well documented.

The split explicit scheme of Gadd (1978a), as implemented in the U.K. Meteorological Office global model, Bell and Dickinson (1987), allows a 30 minute advection and 10 minute adjustment timestep to be used on a 3°×3¾° latitude-longitude grid, with Fourier filtering at high latitudes. The scheme described here was run successfully with this resolution and timestep, using 15 sigma levels in the vertical with a top level about 25mb, and data for September 1987 when the Antarctic polar night jet was at its peak. The maximum zonal wind present in the integration was 112.5 ms^{-1}. The scheme was also run successfully with this timestep using 20 hybrid levels with a top level of 7mb.

A conservation test was carried out using 20 hybrid levels and no physical processes. Moisture was retained as a passive tracer. The global mass-weighted potential temperature decreased by .04% after 10 days and .19% after 50 days, corresponding to a cooling of about .012K/day. The mass-weighted moisture was conserved to within 1 part in 1500 after 50 days. The more accurate moisture conservation was probably due to the use of 32 bit arithmetic in calculating potential temperature differences.

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